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A REMARK ON WEAKLY CONTRACTIVE MAPPINGS

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Abstract. A self-mapping of a metric space with a bounded range is Rakotch contractive if and only if it is weakly contractive in the sense of Alber and Guerre-Delabrière.

1. Introduction

Denote the nonnegative real half-line by $\mathbb{R}_+ := \{r \in \mathbb{R} : r \geq 0\}$ and let (X, d) be a metric space. We begin by recalling the following two definitions.

Definition 1.1. We call a mapping $A : X \to X$ ***weakly contractive*** in the sense of Alber and Guerre-Delabrière [1] if there exists a continuous and increasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\psi^{-1}\{0\} = \{0\} \tag{1.1}$$

and

$$d(Ax, Ay) \leq d(x, y) - \psi(d(x, y)) \quad \text{for all } x, y \in X. \tag{1.2}$$

Definition 1.2. A mapping $A : X \to X$ is called ***Rakotch contractive*** [6] if there exists a decreasing function $\phi : \mathbb{R}_+ \to [0, 1]$ such that

$$\phi(t) < 1 \quad \text{for all } t > 0 \tag{1.3}$$

and

$$d(Ax, Ay) \leq \phi(d(x, y))d(x, y) \quad \text{for all } x, y \in X. \tag{1.4}$$

We are now ready to state our main result. Its proof is given in the next section. Several related issues are discussed in the third and last section of our note. In particular, we review there previous work on both weakly contractive and Rakotch contractive mappings, and explain their significance and relevance for nonlinear analysis.

Theorem 1.3. *Let A be a self-mapping of a metric space with a bounded range. Then A is weakly contractive in the sense of Alber and Guerre-Delabrière if and only if it is Rakotch contractive.*

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2. Proof of Theorem 1.3

Proof. The proof is naturally divided into two parts.

I. Weakly contractive $\Rightarrow$ ***Rakotch contractive***

Let the mapping A satisfy (1.2), where the continuous and increasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies (1.1).

Let $\nu : \mathbb{R}_+ \to \mathbb{R}_+$ be the modulus of continuity of A, that is, let

$$\nu(t) := \sup\{d(Ax, Ay) : x, y \in X,\ d(x, y) \le t\}.$$

It is clear that the function ν is increasing and upper semicontinuous, has a bounded range, and that $\nu(t) \le t$ for all $t \in \mathbb{R}_+$.

Claim 1. $\nu(t) < t$ *for* $t > 0$.

Indeed, suppose that $\nu(s) = s$ for some $s > 0$. Then we could find sequences $(x_n)_{n=1}^{\infty}, (y_n)_{n=1}^{\infty} \subset X$ such that

$$d(x_n, y_n) \le s \quad \text{and} \quad d(Ax_n, Ay_n) \to s,$$

which together with

$$d(Ax_n, Ay_n) \le d(x_n, y_n) - \psi(d(x_n, y_n)) \le d(x_n, y_n) \le s$$

would imply that $d(x_n, y_n) \to s$ and $\psi(d(x_n, y_n)) \to 0$. Thus $\psi(s) = 0$ and by (1.1), $s = 0$, a conclusion which contradicts our choice of $s > 0$. This completes the proof of Claim 1.

Let $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ be the smallest envelope of all the affine majorants of ν, that is, let

$$\omega(t) := \inf\{\alpha(t) : \alpha \text{ is affine}, \alpha(s) \ge \nu(s) \text{ for all } s \in \mathbb{R}_+\}.$$

The function ω is concave and $\nu(t) \le \omega(t) \le t$ for all $t \in \mathbb{R}_+$ by definition.

Claim 2. $\omega(t) < t$ *for* $t > 0$.

Indeed, let s be positive. For $t \ge s$, the real function $t \mapsto t - \nu(t)$ is positive by Claim 1, lower semicontinuous and tends to ∞ as t tends to ∞. Therefore there exists $\epsilon > 0$ such that

$$(*) \qquad t - \nu(t) \ge \epsilon \quad \forall t \ge s.$$

On the other hand, since ν is bounded from above, there exists $r > s$ such that

$$(**) \qquad \nu(t) < r - \epsilon \quad \forall t \ge s.$$

Let $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ be the affine function given by

$$\alpha(t) := \lambda t + (1 - \lambda)(s - \epsilon), \quad \text{where } \lambda = \frac{r - s}{r - s + \epsilon} < 1 \ .$$

The function α is a majorant of ν. To see this, we divide the half-line $\mathbb{R}_+$ into four intervals:

- if $t \in [0, s - \epsilon]$, then

$$\alpha(t) \ge \lambda t + (1 - \lambda)t = t \ge \nu(t);$$

- if $t \in [s-\epsilon, s]$, then

$$\alpha(t) \geq \alpha(s-\epsilon) = s-\epsilon \overset{\textit{by } (*)}{\geq} \nu(s) \geq \nu(t)$$

because both α and ν are increasing functions;
- if $t \in [s, r]$, then

$$\begin{aligned} \alpha(t) &= (t-\epsilon) + [(1-\lambda)(s-\epsilon-t)+\epsilon] \\ &= (t-\epsilon) + \epsilon\frac{r-t}{r-s+\epsilon} \\ &\geq t-\epsilon \\ &\overset{\textit{by } (*)}{\geq} \nu(t); \end{aligned}$$

- if $t \geq r$, then

$$\alpha(t) = s-\epsilon+\lambda(t-s+\epsilon) \geq s-\epsilon+\lambda(r-s+\epsilon) = r-\epsilon \overset{\textit{by } (**)}{>} \nu(t).$$

Thus we see that α is indeed an affine majorant of ν. Hence

$$s > s + (\lambda-1)\epsilon = \alpha(s) \geq \omega(s),$$

as claimed.

Now define the function $\phi : \mathbb{R}_+ \to [0,1]$ by

$$\phi(t) := \frac{\omega(t)}{t} \quad \text{for } t > 0, \ \ \phi(0) := 1.$$

Then $\phi(t) < 1 \ \ \forall t > 0$ by Claim 2, and ϕ is decreasing because

$$\omega(s) \geq \Big(1-\frac{s}{t}\Big)\omega(0) + \frac{s}{t}\omega(t) = \frac{s}{t}\omega(t) \quad \forall\, 0 \leq s < t$$

by the concavity of ω.

Combining all this, we obtain

$$d(Ax, Ay) \leq \nu(d(x,y)) \leq \omega(d(x,y)) = \phi(d(x,y))d(x,y) \quad \forall x, y \in X,$$

which means that A is, in fact, Rakotch contractive, as claimed.

II. Rakotch contractive $\Rightarrow$ weakly contractive

We first assert that when A is Rakotch contractive and has a bounded range, then we can always find a decreasing function $\tilde{\phi} : \mathbb{R}_+ \to [0,1]$ satisfying (1.3) and (1.4) that is also continuous. As a matter of fact, we can use the same construction we employed in the proof of the first implication above. However, defining ν as the modulus of continuity of such a mapping A, that is,

$$\nu(t) := \sup\{d(Ax, Ay) : x, y \in X, \ d(x,y) \leq t\},$$

we prove the analogue of Claim 1 in a slightly different way than before.

Claim $1'$. $\nu(t) < t$ *for* $t > 0$.

If $\nu(s) = s$ for some $s > 0$, we could find sequences $(x_n)_{n=1}^{\infty}, (y_n)_{n=1}^{\infty} \subset X$ satisfying

$$d(x_n, y_n) \leq s \quad \text{and} \quad d(Ax_n, Ay_n) \to s,$$

which when combined with

$$d(Ax_n, Ay_n) \leq \phi(d(x_n, y_n))d(x_n, y_n) \leq d(x_n, y_n) \leq s,$$

would imply that $d(x_n, y_n) \to s > 0$ and $\phi(d(x_n, y_n)) \to 1$, contradicting (1.3). So Claim $1'$ is proved.

From here we can continue by defining ω as the infimum of the affine majorants of ν. The analogue of Claim 2 is then proved just as before, and next we define the function $\tilde{\phi} : \mathbb{R}_+ \to [0, 1]$ that will satisfy all the conditions stated in Definition 1.2 by

$$\tilde{\phi}(t) := \frac{\omega(t)}{t} \quad \text{for } t > 0, \;\; \tilde{\phi}(0) := \lim_{t \to 0^+} \frac{\omega(t)}{t}.$$

Note that continuity away from zero is a consequence of the concavity of ω, while continuity at zero is just by definition.

Thus

$$d(Ax, Ay) \leq \tilde{\phi}(d(x, y))d(x, y).$$

Returning to our assertion, we now note that

$$d(Ax, Ay) \leq \tilde{\phi}(d(x, y))d(x, y) = d(x, y) - (1 - \tilde{\phi}(d(x, y)))d(x, y) \quad \forall x, y \in X.$$

The function $t \mapsto (1 - \tilde{\phi(t)})t$ is the function $\psi(t)$ for which we were looking. □

3. Discussion

In this section we discuss several issues which concern Theorem 1.3 and its proof.

The boundedness assumption regarding the range of A plays a crucial role in Part I (weakly contractive ⇒ Rakotch contractive) of the proof of Theorem 3.1. This is brought out by the following example of a weakly contractive mapping which is not Rakotch contractive (*cf.* [3, page 134]).

Example 3.1. Let $(X, d) := (\mathbb{R}_+, d)$, where

$$d(x, y) := \begin{cases} \max(x, y) & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases} \quad \text{for } x, y \in X,$$

and $A : x \in X \mapsto Ax := \frac{x^2}{1+x} \in X$, an increasing and unbounded function.

Then it is easy to see that

$$d(Ax, Ay) = d(x, y) - \psi(d(x, y)),$$

where $\psi : t \in \mathbb{R}_+ \mapsto \frac{t}{1+t} \in \mathbb{R}_+$ satisfies all the conditions specified in Definition 1.1, so A is weakly contractive.

If there were a function $\phi : \mathbb{R}_+ \to [0, 1]$ satisfying the conditions required in Definition 1.2, then we would have

$$Ay \leq \phi(y)y \quad \forall\, y > 0,$$

or equivalently,

$$\frac{y}{1+y} \leq \phi(y) \quad \forall\, y > 0,$$

which in turn would imply that

$$\frac{y}{1+y} \leq \phi(y) \leq \phi(1) \quad \forall\, y \geq 1.$$

Thus, taking the limit as $y \to \infty$, we would obtain

$$1 \leq \phi(1),$$

which contradicts (1.3).

Returning to the proof of Theorem 1.3, we see that for this specific example, $\nu(t) = \frac{t^2}{1+t} \approx t$ as $t \to \infty$, so $\omega(t) = t$. Hence Claim 2 is not true and we cannot proceed farther.

From the proof of Theorem 1.3 we can infer that when a weakly contractive self-mapping A of a metric space has a bounded range, then the function ψ in the definition of weak contractivity can be chosen to satisfy

$$\psi(t) \to \infty \quad \text{as } t \to \infty. \tag{3.1}$$

Condition (3.1) is, in fact, part of the original definition of weakly contractive mappings as introduced by Alber and Guerre-Delabrière in [1]. Alber's and Guerre-Delabrière's objective in [1] was to extend the classical Banach fixed point theorem for strictly contractive mappings to weakly contractive mappings defined on closed convex subsets of a Hilbert space. They established convergence of sequences generated by various iterative algorithms to the unique fixed point of such a mapping, estimated the rates of convergence, and proved stability of convergence under certain perturbations. Krasnosel'skii *et al.* had also extended the Banach fixed point theorem to a more general class of mappings (see Theorem 3.4 on page 52 of [4]). Weakly contractive mappings in our sense are covered by this result. In this connection see also [9, Theorem 1, page 2684].

Any Rakotch contractive self-mapping of a complete metric space has a unique fixed point and its power iterates converge to this fixed point [6]. As far as we know, this was the first significant generalization of Banach's fixed point theorem. We also recall that most (in the sense of Baire's categories) nonexpansive (that is, 1-Lipschitz) mappings are Rakotch contractive [7]. Moreover, the complement of the set of contractive mappings is, in fact, σ-porous in the space of all nonexpansive mappings [8].

In the proof of our main result we employed some ideas which had already been used in the proof of an extension theorem of Kirszbraun-Valentine type which was established by de Blasi and Pianigiani [2] for the class of contractive mappings defined on compact subsets of a Hilbert space with values in a (possibly different) Hilbert space (see also [5, Theorem 5.2, page 90]). However, the range $[s-\epsilon, s]$ in the proof of Claim 2 above was not covered by the argument presented in [2].

At this point we recall that when (X_1, d_1) and (X_2, d_2) are metric spaces, and D is a subset of (X_1, d_1), then a mapping $A : D \to (X_2, d_2)$ is contractive in the sense of [2] if

$$d_2(Ax, Ay) < d_1(x, y) \quad \text{for all } \ x \neq y \ \text{ in } \ D.$$

The following simple proposition shows that the de Blasi-Pianigiani extension theorem follows from [5, Theorem 5.2] (which concerns extensions of Rakotch contractive mappings in Hilbert space).

Proposition 3.2. *If $D \subset (X_1, d_1)$ is compact and $A : D \to (X_2, d_2)$ satisfies*

$$d_2(Ax, Ay) < d_1(x, y) \ \text{ for all } \ x \neq y \ \text{ in } \ D, \tag{3.2}$$

then A is, in fact, Rakotch contractive.

Proof. Let M be the diameter of D and define the function $\varphi : [0, M] \to [0, 1]$ by $\varphi(0) := 1$ and

$$\varphi(t) := \sup\{d_2(Ax, Ay)/d_1(x, y) : d_1(x, y) \geq t\}$$

for $t > 0$. It is clear that φ is increasing and the mapping A satisfies

$$d_2(Ax, Ay) \leq \varphi(d_1(x, y))d_1(x, y) \ \text{ for all } \ x \text{ and } \ y \ \text{ in } D.$$

To see that $\varphi(t) < 1$ for all $0 < t \leq M$, assume to the contrary that $\varphi(s) = 1$ for some $s > 0$. Then D would contain sequences $\{x_n : n \in \mathbb{N}\}$ and $\{y_n : n \in \mathbb{N}\}$ which converge to x and y, respectively, such that

$$d(x_n, y_n) \geq s > 0 \ \text{ for all } \ n \in \mathbb{N}$$

and

$$d_2(Ax_n, Ay_n)/d_1(x_n, y_n) \to 1 \ \text{ as } \ n \to \infty.$$

Thus $d_2(Ax, Ay) = d_1(x, y)$. The contradiction we have just reached shows that $\varphi(s) < 1$, as claimed. □

This proposition is no longer true if D is not compact. To see this, consider, for instance, the function $f : [1, \infty) \to [1, \infty)$ defined by $f(t) := t + 1/t$.

De Blasi and Pianigiani also showed that a contractive mapping defined on the boundary of a nonempty, open and bounded subset Ω of a Euclidean space $\mathbb{R}^N$ has an extension to all of $\overline{\Omega}$ which has special properties. They used this result to establish existence of solutions to certain vectorial Dirichlet problems. We take this opportunity to note that the de Blasi-Pianigiani result regarding the special properties of extensions can be generalized, by using the same reasoning, to contractive mappings defined on the boundary of a nonempty and relatively open set in a compact subset of a Hilbert space $\mathbb{H}$. The precise statement of this extension is as follows. For subsets S_1 and S_2 of $\mathbb{H}$, we set $d(S_1, S_2) := \inf\{\|x - y\| : x \in S_1, \ y \in S_2\}$.

Theorem 3.3. *Let $\mathbb{H}$ and $\mathbb{L}$ be Hilbert spaces, Ω be a nonempty and relatively open set in a compact subset C of $\mathbb{H}$, and let $A : \partial\Omega \to \mathbb{L}$ be Rakotch contractive. Then there exist a Rakotch contractive extension $B : \overline{\Omega} \to \mathbb{L}$ of A to all of $\overline{\Omega}$, and two sequences $\{\Omega_n\}$ and $\{\lambda_n\}$ with the following properties:*

1. *For each $n \in \mathbb{N}$, $\Omega_n \subset \Omega$ is a nonempty and relatively open subset of C, and $\lambda_n \in [0, 1)$.*

(2) $\overline{\Omega}_n \subset \overline{\Omega}_{n+1}$ *for all* $n \in \mathbb{N}$, $\bigcup_{n\in\mathbb{N}} \Omega_n = \Omega$.
(3) $d(\partial\Omega_n, \partial\Omega_{n+1}) = \rho_n > 0$, $n \in \mathbb{N}$, *and* $\lim_{n\to\infty} \rho_n = 0$.
(4) $\|B(x) - B(y)\| \leq \lambda_n \|x - y\|$ *for every* $x, y \in \overline{\Omega}_n$, $n \in \mathbb{N}$.

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